

QFT C problem sets

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} \text{ J} \cdot \text{s}, c = 2.998 \times 10^8 \text{ m/s}, 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}, \hbar c = 197 \text{ MeV fm}, k_B = 8.617 \times 10^{-5} \text{ eV/K}$$

$$1 \text{ GeV} = 10^3 \text{ MeV} = 10^6 \text{ keV} = 10^9 \text{ eV}, 1 \text{ m} = 100 \text{ cm} = 10^9 \text{ nm} = 10^{15} \text{ fm}$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Contents

1	Dimensional Analysis	2
2	Renormalizable Interactions of a Scalar Field	4
3	Photon Self-Energy	6
4	Photon-Photon (Delbrück) Scattering with Pauli-Villars Regularization	8
5	More on Current Conservation and Ward-Takahashi Identities	12

1 Dimensional Analysis

We consider (in $d = 4$ spacetime dimensions) general Lorentz-invariant interactions of the form

$$\mathcal{L}_{\text{int}} = \sum_i g_i V_i, \quad (1)$$

where g_i is a coupling constant and V_i is a Lorentz scalar operator constructed from fields of different types labeled by f (*e.g.* $f = \phi$ is a scalar or $f = \gamma$ is a photon) and d_i derivatives. Prove that the mass dimension of g_i is given by

$$[g_i] = \Delta_i = 4 - d_i - \sum_f n_{if} (s_f + 1). \quad (2)$$

Here you must use the fact that the propagator $\Delta_f(k)$ of an f -type field scales like

$$\Delta_f(k) \sim \frac{1}{k^{2(1-s_f)}}, \quad (3)$$

where s_f is the spin of the field. An exception important in QED and gravity is that we also effectively have $s_f = 0$ for massless gauge bosons coupled conserved currents.

To do this, recall that $\hbar = 1$. Since $[S] = [\hbar]$, we need S to be dimensionless.

$$S = \int d^d x \mathcal{L}, \quad (4)$$

where $[d^d x] = -d$ requires $[\mathcal{L}] = d$. Thus we have

$$\begin{aligned} [g_i V_i] &= [g_i] + [V_i] = d \\ \Rightarrow [g_i] &= d - [V_i]. \end{aligned} \quad (5)$$

Now V_i is a local operator, so it includes n_{if} fields of type f and d_i derivatives:

$$V_i \sim (\partial_\mu)^{d_i} \prod_f f^{n_{if}}, \quad (6)$$

where $[\partial_\mu] = 1$. Therefore

$$[g_i] = d - d_i [\partial_\mu] - \sum_f n_{if} [f]. \quad (7)$$

The last step is thus to find the dimension of f . This strongly depend on the dimension, so we will focus on $d = 4$. Since

$$\Delta_f(k) \sim \frac{1}{k^{2(1-s_f)}}, \quad (8)$$

we know that

$$\langle \tilde{f}(k) \tilde{f}(0) \rangle \sim \frac{1}{k^{2(1-s_f)}}. \quad (9)$$

Inverse Fourier transformation

$$\int \frac{d^4 k}{(2\pi)^4} e^{ikx} \quad (10)$$

gives

$$\begin{aligned} \langle f(x) f(y) \rangle &\sim \frac{k^4}{k^{2(1-s_f)}} = k^{2(1+s_f)} \\ [f] &= 1 + s_f. \end{aligned} \quad (11)$$

Then we get

$$[g_i] = \Delta_i = 4 - d_i - \sum_f n_{if} (s_f + 1). \quad (12)$$

Additional consideration

Consider a Proca field in $d = 4$ spacetime

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu \quad (13)$$

Since $[\mathcal{L}] = 4$, you get

$$[\partial_\mu A_\nu \partial^\mu A^\nu] = 4 \Leftrightarrow [A_\mu] = 1. \quad (14)$$

Especially, A_μ is a free massive spin 1 field. For this, we observe that there is NO kinetic term for A_0 , so

$$\pi_0 = \frac{\delta \mathcal{L}}{\delta \partial_0 A_0} = 0. \quad (15)$$

Thus A_μ has only 3 independent d.o.f. *i.e.* spin $\{-1, 0, 1\}$. Now propagator can easily be found from the e.o.m.

$$\begin{aligned} \mathcal{L}[A_\mu + \delta A_\mu] &= \mathcal{L}[A_\mu] - \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu) + m^2 A_\mu \delta A^\mu + \mathcal{O}(\delta A^2) \\ &= \mathcal{L}[A_\mu] - \partial_\mu A_\nu \partial^\mu \delta A^\nu + \partial_\nu A_\mu \partial^\mu \delta A^\nu + m^2 A_\mu \delta A^\mu + \mathcal{O}(\delta A^2) \\ &= \mathcal{L}[A_\mu] + (\square A_\mu) \delta A^\mu - (\partial_\mu \partial^\mu A_\mu) \delta A^\mu + m^2 A_\mu \delta A^\mu + \mathcal{O}(\delta A^2) + (\text{total derivative}). \end{aligned} \quad (16)$$

Thus you get

$$\square A_\mu - (\partial_\mu \partial^\mu A_\mu) + m^2 A_\mu = 0. \quad (17)$$

In general, a Green's function needs to obey $\hat{L}G(x, x') = \delta(x - x')$. Then

$$(\square \eta_{\mu\nu} - \partial_\mu \partial_\nu + m^2 \eta_{\mu\nu}) \Delta^\mu = 0. \quad (18)$$

In our case,

$$L_{\mu\nu} G^{\nu\rho} = \delta_\mu^\rho \delta(x - x'), \quad (19)$$

and Fourier transformation gives

$$(-k^2 \eta_{\mu\nu} + k_\mu k_\nu + m^2 \eta_{\mu\nu}) G^{\nu\rho} = \delta_\mu^\rho. \quad (20)$$

We can literally invert this 4×4 matrix, but we can also just use that $G^{\nu\rho}$ must be a tensor

$$G^{\nu\rho} = A(k^2, m^2) \eta^{\nu\rho} + B(k^2, m^2) k^\nu k^\rho. \quad (21)$$

Substituting this into Eq. (20), you get

$$G^{\nu\rho} = \frac{1}{k^2 - m^2} \left(-\eta^{\nu\rho} + \frac{k^\nu k^\rho}{m^2} \right). \quad (22)$$

Clearly, as $k \rightarrow \infty$, $G^{\nu\rho} \sim 1/m^2$. Hence $s_f = 1$.

2 Renormalizable Interactions of a Scalar Field

Consider a single real scalar field ϕ in $d \geq 2$ spacetime dimensions. Do not impose any symmetries other than Lorentz invariance. Enumerate all renormalizable interactions for such a field. Which ones are marginal and which ones are relevant? For which values of d are there healthy non-trivial interactions? Based on the outcome of this problem, it is a surprising and interesting fact that string theory predicts the existence of healthy interacting QFTs in $d = 5, 6$ spacetime dimensions.

A real scalar field always has mass dimension determined by its kinetic term

$$\left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right] = d \Leftrightarrow [\phi] = \frac{d-2}{2}. \quad (23)$$

Using the formula derived beforehand

$$\begin{aligned} \Delta_i &= d - d_i - n_\phi \frac{d-2}{2} \\ &= d \left(1 - \frac{n_\phi}{2} \right) - d_i + n_\phi. \end{aligned} \quad (24)$$

Recall that for renormalizability we need $\Delta_i \geq 0$.

(i) $d = 2$

$$\begin{aligned} \Delta_i &= 2 \left(1 - \frac{n_\phi}{2} \right) - d_i + n_\phi \\ &= 2 - d_i, \end{aligned} \quad (25)$$

then

- Only 2-derivative interactions are renormalizable
- All polynomials in ϕ (*i.e.* ϕ^n) are renormalizable, even if they include 2-derivatives (*i.e.* $\partial^2 \phi^n$).

(ii) $d = 3$

$$\begin{aligned} \Delta_i &= 3 \left(1 - \frac{n_\phi}{2} \right) - d_i + n_\phi \\ &= 3 - d_i - \frac{n_\phi}{2}. \end{aligned} \quad (26)$$

- For an interaction term, we require $n_\phi > 2$ since it is free otherwise
- We now need $d_i = 0$ because $d_i \in 2\mathbb{Z}$ is necessary for Lorentz invariance and $n_\phi > 2$

Thus we get

$$\Delta_i = 3 - \frac{n_\phi}{2} \Rightarrow n_\phi \leq 6. \quad (27)$$

- All polynomials up to ϕ^6 with no derivatives are allowed

(iii) $d = 4$

$$\begin{aligned} \Delta_i &= 4 \left(1 - \frac{n_\phi}{2} \right) - d_i + n_\phi \\ &= 4 - d_i - n_\phi. \end{aligned} \quad (28)$$

- Same argument for noderivative interactions as in $d = 3$ applies here
- $d_i = 0$ gives $n_\phi \leq 4$. Then all polynomials up to ϕ^4 with no derivatives are allowed

(iv) $d = 5$

$$\begin{aligned}\Delta_i &= 5 \left(1 - \frac{n_\phi}{2}\right) - d_i + n_\phi \\ &= 5 - d_i - \frac{3n_\phi}{2}.\end{aligned}\tag{29}$$

- Same argument for noderivative interactions as in $d = 3$ applies here, and it gets worse!
- $d_i = 0$ gives $n_\phi \leq 10/3$. Then ϕ^3 with no derivatives is still allowed
- Although this interaction is perturbatively okay, the cubic potential does not have a lower energy bound. This theory leads to unhealthy phenomenology.

(v) $d = 6$

- From exactly the same argument as $d = 5$ case, ϕ^3 with no derivatives is still allowed but not healthy

(vi) $d > 6$

$$\begin{aligned}\Delta_i &= d \left(1 - \frac{n_\phi}{2}\right) - d_i + n_\phi \\ &\leq d - d_i - 3 \frac{d-2}{2} \quad (\because n_\phi \geq 3) \\ &= -\frac{d}{2} + 1 - d_i \\ &< 0,\end{aligned}\tag{30}$$

then NO renormalizable interactions for $d > 6$.

3 Photon Self-Energy

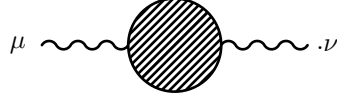
Consider the 1PI photon self-energy function in QED. Bose symmetry and Lorentz invariance imply the general form

$$\Pi_{\mu\nu}(k) = A(k^2) \eta_{\mu\nu} + B(k^2) k_\mu k_\nu, \quad (31)$$

with $A(k^2), B(k^2)$ general Lorentz-scalar functions of l^2 called structure functions.

(a) What does the Ward-Takahashi identity imply about the structure functions? Equivalently, current conservation or transversality of physical massless photons – all loosely referred to as gauge invariance.

$\Pi_{\mu\nu}(k)$ is related to


(32)

So it is $\langle j^\mu(x) j^\nu(y) \rangle$ as these have to couple to the external photons! Then from Ward-Takahashi identity, we have

$$\partial_\mu \langle j^\mu(x) j^\nu(y) \rangle = 0 = \partial_\nu \langle j^\mu(x) j^\nu(y) \rangle. \quad (33)$$

Since j^μ is unchanged

$$k^\mu \Pi_{\mu\nu}(k) = k^\nu \Pi_{\mu\nu}(k) \Rightarrow k_\nu A(k^2) + B(k^2) k^2 k_\nu = 0. \quad (34)$$

Thus we get

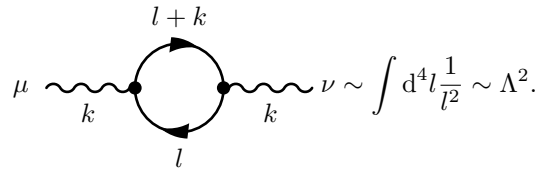
$$A(k^2) = -B(k^2) k^2. \quad (35)$$

(b) Consider the 1-loop correction to $\Pi_{\mu\nu}(k)$ coming from one electron loop. Use the fact that this has degree of divergence $D = 2$ to argue that the expected divergences are

$$c_1 \Lambda^2 g_{\mu\nu} + \log(\Lambda) (c_2 k^2 g_{\mu\nu} + c_3 k_\mu k_\nu), \quad (36)$$

with c_1, c_2, c_3 dimensionless constants and Λ a schematic momentum cutoff. Assuming a cutoff scheme that preserves the Ward-Takahashi identity, what can you say about these constants? Verify this explicitly by computing the divergent part of the diagram in dimensional regularization, which does preserve the Ward-Takahashi identity.

What we expect is


(37)

Then we can have three possible divergences naively

$$\Lambda^2, \quad \Lambda, \quad \log \Lambda. \quad (38)$$

By dimensional analysis, the divergences need to go as Λ^2 , so

$$A_{\mu\nu} \Lambda^2 + B_{\mu\nu} \Lambda + \log(\Lambda) C_{\mu\nu}. \quad (39)$$

$A_{\mu\nu}, B_{\mu\nu}$ and $C_{\mu\nu}$ need to be symmetric and can only be built from k_μ and $\eta_{\mu\nu}$, so we just have $\eta_{\mu\nu}, k_\mu k_\nu$. Dimensional analysis also implies no polynomial in k^2 , giving rise to a linear divergence. Since we need our counterterms to be local, we cannot allow for things like $\sqrt{k^2}$. Thus we get

$$c_1 \Lambda^2 \eta_{\mu\nu} + 0 \cdot \Lambda + \log(\Lambda) (c_2 k^2 \eta_{\mu\nu} + c_3 k_\mu k_\nu). \quad (40)$$

If the cutoff schemes preserve the Ward-Takahashi identity, then

$$k^\mu c_1 \Lambda^2 \eta_{\mu\nu} = 0 \Rightarrow c_1 = 0, \quad (41)$$

$$\log(\Lambda) (c_2 k^2 k_\nu + c_3 k^2 k_\nu) = 0 \Rightarrow c_2 = -c_3. \quad (42)$$

(c) Now compute the most divergent constant c_1 using a hard Euclidean loop-momentum cutoff $k_E < \Lambda$. We compute the diagram as always

$$\begin{aligned}
\frac{1}{e^2} \left[\mu \text{---}\text{wavy}(k)\text{---}\bullet \circlearrowright^{l+k}_l \bullet \text{---}\text{wavy}(k)\text{---}\nu \right] &= - \int \frac{d^4 l}{(2\pi)^4} \text{Tr} \left[\gamma^\nu \frac{i}{\not{l}-m} \gamma^\mu \frac{i}{\not{l}+\not{k}-m} \right] \\
&= \int \frac{d^4 l}{(2\pi)^4} \text{Tr} \left[\gamma^\nu \frac{\gamma^\rho l_\rho + m}{l^2 - m^2} \gamma^\mu \frac{\gamma^\sigma (l+k)_\sigma + m}{(l+k)^2 - m^2} \right] \\
&= \int \frac{d^4 l}{(2\pi)^4} \text{Tr} [\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma] \frac{l_\rho}{l^2 - m^2} \frac{(l+k)_\sigma}{(l+k)^2 - m^2} \\
&\quad + \int \frac{d^4 l}{(2\pi)^4} \text{Tr} [\gamma^\nu \gamma^\mu] \frac{m}{l^2 - m^2} \frac{m}{(l+k)^2 - m^2}.
\end{aligned} \tag{43}$$

For the first integral, we get

$$\begin{aligned} \int \frac{d^4 l}{(2\pi)^4} \text{Tr} [\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma] \frac{l_\rho}{l^2 - m^2} \frac{(l+k)_\sigma}{(l+k)^2 - m^2} &= \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \text{Tr} [\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma] \frac{l_\rho (l+k)_\sigma}{\left\{ (l+k\alpha)^2 - [m^2 + k^2\alpha(\alpha-1)] \right\}^2} \\ &= \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma] \frac{(q-k\alpha)_\rho (q-k\alpha+k)_\sigma}{(q^2 - D + i\epsilon)^2}. \end{aligned} \quad (44)$$

Now, we want to single out the most divergent term, so the term with q^2 in the numerator

$$\begin{aligned}
\frac{1}{e^2} c_1 &\sim \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma] \frac{q_\rho q_\sigma}{(q^2 - D + i\epsilon)^2} \\
&= \frac{1}{4} \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma] \frac{q^2 \eta_{\rho\sigma}}{(q^2 - D + i\epsilon)^2} \quad \left(q_\rho q_\sigma \rightarrow \frac{1}{4} q^2 \eta_{\rho\sigma} \text{ in the loop integral} \right) \\
&= -\frac{i}{4} \int_0^1 dx \int \frac{d^4 q_E}{(2\pi)^4} \text{Tr} [\gamma^\nu (2\eta^{\rho\mu} - \gamma^\mu \gamma^\rho) \gamma^\sigma \eta_{\rho\sigma}] \frac{q_E^2}{(q_E^2 + D)^2} \quad (\text{Now } q^0 \rightarrow iq^0) \\
&= \frac{i}{2} \int_0^1 dx \int_0^\Lambda dq_E \text{Tr} [\gamma^\mu \gamma^\nu] \frac{q_E^5}{(q_E^2 + D)^2} \frac{1}{(2\pi)^4} \int d\Omega_3 \\
&= \frac{4i\eta^{\mu\nu}}{24\pi^3} \int_0^1 dx \int_0^\Lambda dq_E \frac{q_E^5}{(q_E^2 + D)^2} \quad \left(\int d\Omega_3 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2, \quad \text{Tr} [\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu} \right) \\
&= \frac{4i\eta^{\mu\nu}}{24\pi^3} \int_0^1 dx \left[\underbrace{\frac{q_E^2}{2}}_{\text{Quadratic}} - \underbrace{\frac{D^2}{2(D + q_E^2)}}_{\rightarrow 0 \text{ as } \Lambda \rightarrow \infty} - \underbrace{A \log(D + q_E^2)}_{\text{Log}} \right] \Bigg|_0^\Lambda. \quad (\text{Mathematica}) \tag{45}
\end{aligned}$$

Thus the most divergent term in c_1 is given by

$$\frac{ie^2}{(2\pi)^3}\Lambda^2\eta^{\mu\nu}. \quad (46)$$

The fact that $c_1 \neq 0$ implies a quadratically divergent renormalization of the photon mass term

$$m_\gamma^2 A^\mu A_\mu. \quad (47)$$

This term is not gauge invariant, but neither was our momentum cutoff Λ . We can remove the divergence and explicitly restore the Ward-Takahashi identity by fine-tuning m_γ^2 . The lesson is that working with a gauge non-invariant cutoff requires gauge non-invariant counterterms. Needless to say this is cumbersome and we will avoid it when possible; but it is not a problem of principle.

4 Photon-Photon (Delbrück) Scattering with Pauli-Villars Regularization

Consider photon-photon scattering at 1-loop order in QED.

(a) Show that this has degree of divergence $D = 0$ and is thus superficially log-divergent. Use this to explicitly parametrize the expected form of the divergence, as you did in the photon self-energy. Show explicitly that the Ward-Takahashi identity implies that the divergent piece is absent. So if we calculate with a gauge invariant regulator, photon scattering is a finite process. Indeed, a log-divergent photon scattering diagram would require a marginal, gauge non-invariant counterterm

$$(A^\mu A_\mu)^2 \quad (48)$$

in the Lagrangian to restore the Ward-Takahashi identity.

We are thinking about a four point vertex function

$$V_{4\mu\nu\rho\sigma} = \text{Diagram} \quad (49)$$

At 1-loop, this includes

$$\text{Diagram} \sim \int d^4l \left(\frac{1}{l}\right)^4 \sim \log \Lambda \quad (50)$$

This is superficially logarithmically diverging, so its divergence structure should just be

$$c_1 \log(\Lambda) \underbrace{(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})}_{\text{symmetrized}}. \quad (51)$$

From Ward-Takahashi identity, we have

$$k_1^\mu V_{4\mu\nu\rho\sigma} = 0 \Rightarrow c_1 = 0. \quad (52)$$

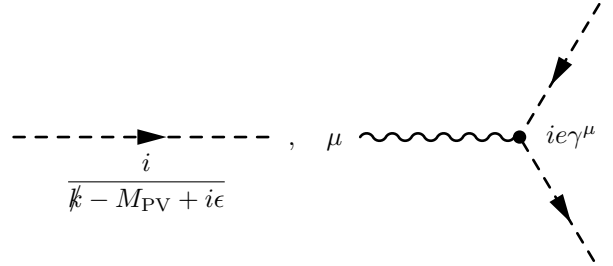
Thus, the divergent piece in this scattering is absent.

(b) To verify this prediction, calculate these superficially divergent terms using a gauge-invariant regulator and show that they add up to zero. (How many diagrams are there?) Rather than using dimensional regularization (of course you can but there is subtlety in spinor representation), you will use another useful gauge-invariant method called Pauli-Villars regularization. This involves taking every closed electron loop (with electron mass m) and subtracting from it exactly the same function of momenta but with the electron mass m replaced by some large mass scale $M_{\text{PV}} \sim \Lambda$ that acts as the cutoff scale. Physically, this recipe is the result of adding a new very heavy spinor field Ψ_{PV} of mass M_{PV} (the Pauli-Villars regulator field). It has the same Lagrangian as the electron (including the coupling to A_μ , but with $m \rightarrow M_{\text{PV}}$). However, it is a bosonic field, hence it gives a relative minus sign in closed loops. This means that it violates the spin-statistics theorem (it is a bosonic spinor “ghost” field and violates unitarity), but it preserves Lorentz and gauge invariance. This makes it a useful gauge-invariant regulator. In the log-divergent photon scattering, a single Pauli-Villars regulator field suffices. To regulate the quadratically divergent photon self-energy, one needs to add three different regulator fields with different statistics. Pauli-Villars are another U(1) gauge invariant regulator, which will prove highly useful in the chiral anomaly computation. A nice description is given in Schwartz Appendix B.

Let us add a term

$$\Delta\mathcal{L}_{\text{PV}} = \bar{\Psi}_{\text{PV}} (i\cancel{\partial} - e\cancel{A} - M_{\text{PV}}) \Psi_{\text{PV}}, \quad (53)$$

and this gives rise to



The diagram shows a horizontal dashed line with an arrow pointing right, labeled with i above and $\frac{i}{\cancel{k} - M_{\text{PV}} + i\epsilon}$ below. To its right is a vertex where a wavy line (photon) enters from the left, labeled μ , and two dashed lines exit to the right. The vertex is labeled $ie\gamma^\mu$. The entire expression is followed by a comma and the equation number (54).

and a (+1) (relative minus 1) in a closed loop. As before, for the four point amplitude, we have a good amount of diagrams, which we will quote honestly.

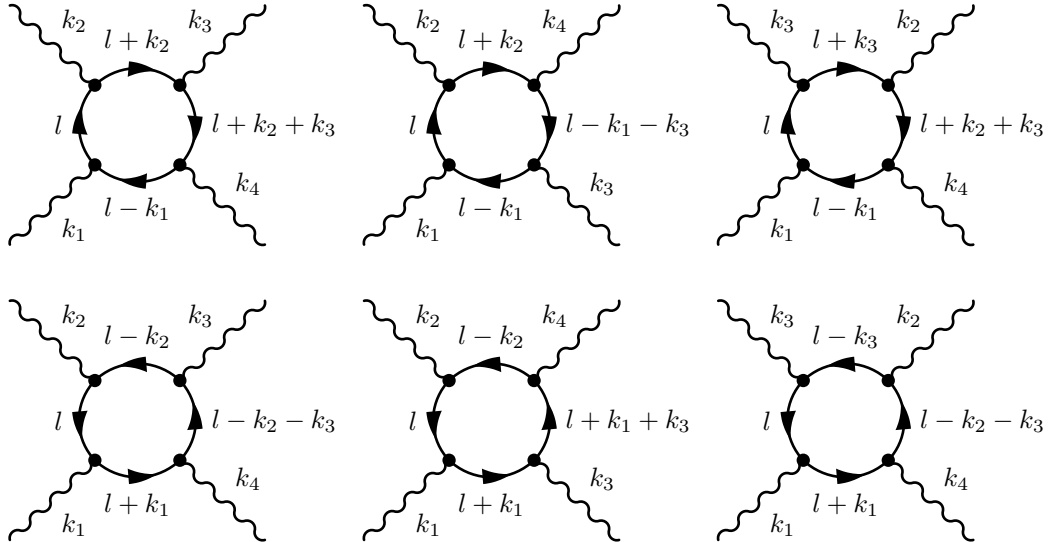
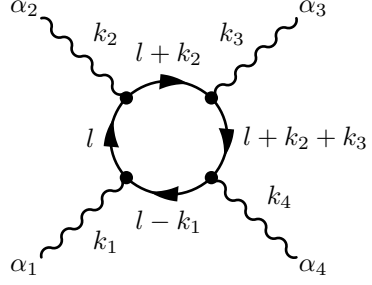


Figure 1: All diagrams contributing to the 1-loop correction

All of these diagrams will also be there but with the Pauli-Villars particle running in the loop. As before, we will focus on one diagram and only single out the most divergent term. Therefore, we multiply all propagators by

$$\frac{\cancel{k} + m}{\cancel{k} + M_{\text{PV}}}, \quad (55)$$

and use Feynman parameters and single out the quartic term in the numerator such that



$$\begin{aligned}
& \simeq -6e^4 \int dF_4 \int \frac{d^4 q}{(2\pi)^4} \frac{\text{Tr} [\not{q} \gamma^{\alpha_1} \not{q} \gamma^{\alpha_2} \not{q} \gamma^{\alpha_3} \not{q} \gamma^{\alpha_4}]}{(q^2 - D_{m_e})^4} \\
& + 6e^4 \int dF_4 \int \frac{d^4 q}{(2\pi)^4} \frac{\text{Tr} [\not{q} \gamma^{\alpha_1} \not{q} \gamma^{\alpha_2} \not{q} \gamma^{\alpha_3} \not{q} \gamma^{\alpha_4}]}{(q^2 - D_{M_{\text{PV}}})^4}.
\end{aligned} \tag{56}$$

Now let us first do some Dirac matrix algebra

$$\text{Tr} [\not{q} \gamma^{\alpha_1} \not{q} \gamma^{\alpha_2} \not{q} \gamma^{\alpha_3} \not{q} \gamma^{\alpha_4}] = q_{\beta_1} q_{\beta_2} q_{\beta_3} q_{\beta_4} \text{Tr} [\gamma^{\beta_1} \gamma^{\alpha_1} \gamma^{\beta_2} \gamma^{\alpha_2} \gamma^{\beta_3} \gamma^{\alpha_3} \gamma^{\beta_4} \gamma^{\alpha_4}]. \tag{57}$$

Observe that, if q_i is not contracted, we are again have an integral like

$$\int dq_1 dq_2 q_1 q_2 f(q^2). \tag{58}$$

Thus, it would vanish. Now there are 3 combinations of contracting

$$\tilde{C} (\eta_{\beta_1 \beta_2} \eta_{\beta_3 \beta_4} + \eta_{\beta_1 \beta_3} \eta_{\beta_2 \beta_4} + \eta_{\beta_1 \beta_4} \eta_{\beta_2 \beta_3}) q^4, \tag{59}$$

so we just need to figure out the coefficient $\tilde{C}$. For this, contract by $\eta^{\beta_1 \beta_2} \eta^{\beta_3 \beta_4}$ to get

$$\tilde{C} (4 \cdot 4 + 4 + 4) q^4 = 24 \tilde{C} q^4 \Rightarrow \tilde{C} = \frac{1}{24}. \tag{60}$$

Then

$$\begin{aligned}
\text{Tr} [\not{q} \gamma^{\alpha_1} \not{q} \gamma^{\alpha_2} \not{q} \gamma^{\alpha_3} \not{q} \gamma^{\alpha_4}] &= \frac{q^4}{24} (\eta_{\beta_1 \beta_2} \eta_{\beta_3 \beta_4} + \eta_{\beta_1 \beta_3} \eta_{\beta_2 \beta_4} + \eta_{\beta_1 \beta_4} \eta_{\beta_2 \beta_3}) \text{Tr} [\gamma^{\beta_1} \gamma^{\alpha_1} \gamma^{\beta_2} \gamma^{\alpha_2} \gamma^{\beta_3} \gamma^{\alpha_3} \gamma^{\beta_4} \gamma^{\alpha_4}] \\
&= \frac{q^4}{3} [\text{Tr} (\gamma^{\alpha_1} \gamma^{\alpha_2} \gamma^{\alpha_3} \gamma^{\alpha_4}) - \eta^{\alpha_1 \alpha_3} \text{Tr} (\gamma^{\alpha_2} \gamma^{\alpha_4})] \\
&= \frac{4q^4}{3} (\eta^{\alpha_1 \alpha_2} \eta^{\alpha_3 \alpha_4} - 2\eta^{\alpha_1 \alpha_3} \eta^{\alpha_2 \alpha_4} + \eta^{\alpha_1 \alpha_4} \eta^{\alpha_2 \alpha_3}).
\end{aligned} \tag{61}$$

Interestingly, we see that

$$\begin{aligned}
& \text{Tr} [\not{q} \gamma^{\alpha_1} \not{q} \gamma^{\alpha_2} \not{q} \gamma^{\alpha_3} \not{q} \gamma^{\alpha_4}] + (\alpha_3 \leftrightarrow \alpha_4) + (\alpha_2 \leftrightarrow \alpha_3) \\
&= \frac{4q^4}{3} \left[(\eta^{\alpha_1 \alpha_2} \eta^{\alpha_3 \alpha_4} - 2\eta^{\alpha_1 \alpha_3} \eta^{\alpha_2 \alpha_4} + \eta^{\alpha_1 \alpha_4} \eta^{\alpha_2 \alpha_3}) \right. \\
&\quad + (\eta^{\alpha_1 \alpha_2} \eta^{\alpha_4 \alpha_3} - 2\eta^{\alpha_1 \alpha_4} \eta^{\alpha_2 \alpha_3} + \eta^{\alpha_1 \alpha_3} \eta^{\alpha_2 \alpha_4}) \\
&\quad \left. + (\eta^{\alpha_1 \alpha_3} \eta^{\alpha_2 \alpha_4} - 2\eta^{\alpha_1 \alpha_2} \eta^{\alpha_3 \alpha_4} + \eta^{\alpha_1 \alpha_4} \eta^{\alpha_3 \alpha_2}) \right] \\
&= 0.
\end{aligned} \tag{62}$$

Thus, when the denominators of the upper 3 diagrams are identical, we can conclude that all these 3 diagrams cancel. For this, we need to check that the D in the denominator is identical. Let us compute D for diagrams generally. For now, I will just call the momenta

$$\{q, q + k_i\}, \tag{63}$$

where k_i is built out of external momenta and $i \in \{1, 2, 3\}$. Then

$$\begin{aligned}
& x_1 \left[(q + k_1)^2 - m^2 \right] + x_2 \left[(q + k_2)^2 - m^2 \right] + x_3 \left[(q + k_3)^2 - m^2 \right] + (1 - x_1 - x_2 - x_3) (q^2 - m^2) \\
&= (q + x_1 k_1 + x_2 k_2 + x_3 k_3)^2 - \sum_{i=1}^3 (x_i^2 - x_i) k_i^2 - m^2 \\
&\Rightarrow D = m^2 + \sum_{i=1}^3 x_i (x_i - 1) k_i^2.
\end{aligned} \tag{64}$$

We observe that the label i is completely arbitrary, since Feynman parameters can be distributed. Sadly, all of these D 's are different, so we actually need to integrate a bit. Before that, an identical game can be played for the lower three diagrams, since they can simply be found by exchanging the indices α_1 is connected to with respect to the upper diagrams.

$$\begin{aligned}
& \int dF_4 \int \frac{d^4 q}{(2\pi)^4} \left[\frac{q^4}{(q^2 - D_{m_e})^4} - \frac{q^4}{(q^2 - D_{M_{PV}})^4} \right] \\
& \propto i \int dF_4 \int_0^\infty dq_E \left[\frac{q_E^7}{(q_E^2 + D_{m_e})^4} - \frac{q_E^7}{(q_E^2 + D_{M_{PV}})^4} \right] \quad (\text{Wick rotation}) \\
&= i \int dF_4 \frac{1}{12} \left[\frac{D_{m_e} (11 D_{m_e}^2 + 27 D_{m_e} q^2 + 18 q^4)}{(D_{m_e} + q^2)^3} + 6 \log(D_{m_e} + q^2) - (D_{m_e} \leftrightarrow D_{M_{PV}}) \right] \Big|_0^\infty \\
&= \frac{i}{2} \int dF_4 \log \left(\frac{D_{m_e} + q^2}{D_{M_{PV}} + q^2} \right) \Big|_0^\infty \\
&= \frac{i}{2} \int dF_4 \log \left(\frac{D_{M_{PV}}}{D_{m_e}} \right).
\end{aligned} \tag{65}$$

For the integrand

$$\begin{aligned}
\log \left(\frac{D_{M_{PV}}}{D_{m_e}} \right) &= \log \left[\frac{M_{PV}^2 + \sum_{i=1}^3 x_i (x_i - 1) k_i^2}{m_e^2 + \sum_{i=1}^3 x_i (x_i - 1) k_i^2} \right] \\
&\equiv \log \left(\frac{M_{PV}^2}{m_e^2} \right) + \log \left(\frac{1 + A/M_{PV}^2}{1 + A/m_e^2} \right) \\
&= \log \left(\frac{M_{PV}^2}{m_e^2} \right) + \frac{A}{M_{PV}^2} + \mathcal{O} \left(\frac{1}{M_{PV}^4} \right) + (\text{Finite}),
\end{aligned} \tag{66}$$

then the logarithmically divergent term does not depend on $A \equiv \sum_{i=1}^3 x_i (x_i - 1) k_i^2$. Thus the divergences cancel!

Observe that a cutoff regulator would have given

$$\log \left(1 + \frac{\Lambda^2}{D} \right) = \log \left(\frac{\Lambda^2}{D} \right) + (\text{Finite}), \tag{67}$$

then it would depend on D and the divergence would not vanish. Therefore, cutoff regularization is not gauge-invariant.

5 More on Current Conservation and Ward-Takahashi Identities

These notions are often first discussed in the context of gauge invariance but they are really properties of global (*i.e.* non-gauge) symmetries. For simplicity, consider a model with a global U(1) flavor symmetry that contains a complex scalar field Φ and a Dirac field ψ . (There are no gauge fields.) The free part of the Lagrangian is

$$\mathcal{L}_{\text{free}} = \partial^\mu \bar{\Phi} \partial_\mu \Phi - m^2 |\Phi|^2 + \bar{\psi} (i \not{\partial} - M) \psi. \quad (68)$$

(a) Assume that Φ has charge q_Φ and ψ has charge q_ψ under U(1) symmetry. Spell out the action of the symmetry on the fields. Then, use Noether's theorem to find the corresponding conserved current j_μ and check explicitly that it is conserved on shell (*i.e.* using the equation of motion). Show that the form of j_μ and its properties are not modified in the presence of (non-derivative) interactions that preserve the U(1) symmetry. Check explicitly that

$$[j^0(t, \mathbf{x}), \mathcal{O}(t, \mathbf{y})] = -q_\mathcal{O} \mathcal{O}(t, \mathbf{y}) \delta^3(\mathbf{x} - \mathbf{y}), \quad (69)$$

where $\mathcal{O}$ is any local operator constructed out of the fields and their derivatives and $q_\mathcal{O}$ is its U(1) charge. (Hint: first show it for the fundamental fields themselves.) It is standard to define the conserved charge Q by the following integral,

$$Q \equiv \int d^3\mathbf{x} j^0(0, \mathbf{x}). \quad (70)$$

Note that this integral may diverge (In the IR, at long distances, as $|\mathbf{x}| \rightarrow \infty$), so that Q may not exist as a well-defined operator. Conclude that

$$[Q, \mathcal{O}(t, \mathbf{x})] = -q_\mathcal{O} \mathcal{O}(t, \mathbf{x}). \quad (71)$$

Thus the Hermitian charge operator Q measures the U(1) charge of the operators. We also say it generates the U(1) transformations. Argue that this formula makes sense even if the charge operator Q does not formally exist, *i.e.* unlike Q itself the commutator is not afflicted by any divergences.

(b) Establish the Ward-Takahashi identity for the global U(1) symmetry current j_μ ,

$$\partial_\mu \langle T j^\mu(x) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = - \sum_{k=1}^n q_k \delta^4(x - x_k) \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle. \quad (72)$$

Here $T \cdots$ denotes time ordering and q_k is the U(1) charge of $\mathcal{O}_k$. Assuming the left-hand side is indeed (as naively appears to be) a total derivative, show that the correlation function

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \quad (73)$$

can only be non-zero if the U(1) charges of the operators $\mathcal{O}_k$ add up to zero.

(c) Now consider conventional QED: there is no scalar field Φ , but there is an electron ψ of charge $q_\psi = -e$. There is now also a dynamical photon field A_μ that couples to the electron. A nice feature of this theory is that the interaction is linear in A_μ , *i.e.*

$$\mathcal{L}_{\text{int}} = A_\mu j^\mu = -e A_\mu \bar{\psi} \gamma^\mu \psi. \quad (74)$$

Use this to argue that adding a single on-shell photon of momentum q^μ (with $q^2 = 0$) and transverse polarization $\varepsilon^\mu(q)$ to any scattering process $S(\alpha \rightarrow \beta) = \langle \beta, \text{out} | \alpha, \text{in} \rangle$ gives an amplitude proportional to

$$\mathcal{A}(q, \varepsilon, \alpha, \beta) \sim \varepsilon^\mu(q) \int d^4x e^{iqx} \langle \beta, \text{out} | j_\mu(x) | \alpha, \text{in} \rangle. \quad (75)$$

(The argument can be generalized to many photons; see Weinberg section 10.5. Depending on the scheme we use, the exact formula may also involve Z -factors from the LSZ reduction formula which do not play a role in the argument here.)

For concreteness, you can assume that the in and out states are obtained by acting with some operators $\mathcal{O}_k(x_k)$ on the vacuum. Show that the Ward-Takahashi identity implies that this on-shell S-matrix element does not depend on any longitudinal component $\sim q^\mu$ in the photon polarization $\varepsilon^\mu(q)$. This is sometimes loosely referred to as the “gauge invariance” of the S-matrix, since shifting $\varepsilon_\mu(q)$ by $iq_\mu \lambda(q)$ can be thought of as arising from a gauge transformation $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x)$. It ensures that the amplitude only depends on the two physical, transverse photon polarization. A similar argument shows why we can always drop any terms $\sim k_\mu$ or $\sim k_\nu$ in the photon propagator.